

MR2732572 (2011m:53187) 53D55 (37J35 81R12 81S10)

Garay, Mauricio D. (D-MPI); van Straten, Duco (D-MNZ-IM)

Classical and quantum integrability. (English, Russian summaries)

Mosc. Math. J. **10** (2010), no. 3, 519–545, 661.

This paper provides a concrete criterion for a Liouville integrable Hamiltonian system to be quantizable, i.e. to be the classical limit of a quantum integrable system. Given a Liouville integrable Hamiltonian system, i.e. a family of n Poisson-commuting independent polynomials $f_j: T^*\mathbb{C}^n \simeq \mathbb{C}^{2n} \rightarrow \mathbb{C}$ for $j = 1, \dots, n$, the authors attach to $f = (f_1, \dots, f_n)$ a complex $C_f^\bullet$ by using a standard (Koszul) algebraic procedure; then they prove that the quantizability problem (i.e. the existence of n commuting $\hbar$ -differential operators F_j whose principal symbols are the given f_j 's) can be carried out provided that certain “anomaly” classes in the cohomology module $H^2(C_f^\bullet)$ vanish. Sufficient topological conditions on the map f in order that this vanishing holds are also pointed out.

Reviewed by [Corrado Marastoni](#)

References

1. V. I. Arnold, *A theorem of Liouville concerning integrable problems of dynamics*, Sibirsk. Mat. Zh. **4** (1963), 471–474 (Russian). MR 0147742 [MR0147742 \(26 #5256\)](#)
2. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Quantum mechanics as a deformation of classical mechanics*, Lett. Math. Phys. **1** (1975/77), no. 6, 521–530. MR 0674337 [MR0674337 \(58 #32617\)](#)
3. M. Born, W. Heisenberg, and P. Jordan, *Zur Quantenmechanik II*, Z. Phys. **35** (1926), 557–615.
4. A. D'Agnolo and P. Schapira, *Quantization of complex Lagrangian submanifolds*, Adv. Math. **213** (2007), no. 1, 358–379. MR 2331247 [MR2331247 \(2009d:32005\)](#)
5. P. Deligne, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970. MR 0417174 [MR0417174 \(54 #5232\)](#)
6. P. Deligne, *Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte*, Selecta Math. (N.S.) **1** (1995), no. 4, 667–697. MR 1383583 [MR1383583 \(97a:58068\)](#)
7. P. A. M. Dirac, *The fundamental equations of quantum mechanics*, Proc. Roy. Soc. A **109** (1926), 642–653.
8. B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), no. 2, 213–238. MR 1293654 [MR1293654 \(95h:58062\)](#)
9. B. V. Fedosov, *Deformation quantization and index theory*, Mathematical Topics, vol. 9, Akademie Verlag, Berlin, 1996. MR 1376365 [MR1376365 \(97a:58179\)](#)
10. M. D. Garay, *A rigidity theorem for Lagrangian deformations*, Compos. Math. **141** (2005), no. 6, 1602–1614. MR 2188452 [MR2188452 \(2006k:32057\)](#)

11. A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 95–103. MR 0199194 [MR0199194 \(33 #7343\)](#)
12. A. Grothendieck, *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Exposé III: Morphismes lisses: propriétés de prolongement. Lecture Notes in Mathematics, vol. 224, pp. 58–86, Springer-Verlag, Berlin, 1971. MR 0354651. [MR0354651 \(50 #7129\)](#)
13. A. Grothendieck, *Géométrie formelle et géométrie algébrique*, Séminaire Bourbaki, Vol. 5, Soc. Math. France, Paris, 1995, Exp. No. 182, 193–220, errata p. 390. MR 1603467 [MR1603467](#)
14. J. Hietarinta, *Quantum integrability is not a trivial consequence of classical integrability*, Phys. Lett. A **93** (1982/83), no. 2, 55–57. MR 687495 [MR0687495 \(85b:81084\)](#)
15. H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203, 205–326. MR 0199184 [MR0199184 \(33 #7333\)](#)
16. M. Kontsevich, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. **66** (2003), no. 3, 157–216. MR 2062626 [MR2062626 \(2005i:53122\)](#)
17. J. Liouville, *Note sur l'intégration des équations différentielles de la dynamique*, J. Math. Pure Appl. **20** (1855), 137–138.
18. B. Malgrange, *Sur les points singuliers des équations différentielles*, Enseignement Math. (2) **20** (1974), 147–176. MR 0368074 [MR0368074 \(51 #4316\)](#)
19. H. Matsumura, *Commutative algebra*, W. A. Benjamin, Inc., New York, 1970. MR 0266911 [MR0266911 \(42 #1813\)](#)
20. H. Mineur, *Réduction des systèmes mécaniques à n degrés de liberté admettant n intégrales premières uniformes en involution aux systèmes à variables séparées*, J. Math. Pure Appl. **15** (1936), 221–267.
21. J. E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Cambridge Philos. Soc. **45** (1949), 99–124. MR 0029330 [MR0029330 \(10,582d\)](#)
22. R. Nest and B. Tsygan, *Algebraic index theorem for families*, Adv. Math. **113** (1995), no. 2, 151–205. MR 1337107 [MR1337107 \(96j:58163a\)](#)
23. N. Reshetikhin and M. Yakimov, *Deformation quantization of Lagrangian fiber bundles*, Conférence Moshé Flato 1999, Vol. II (Dijon), Math. Phys. Stud., vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, pp. 263–287. MR 1805921 [MR1805921 \(2002h:53156\)](#)
24. M. Robnik, *A note concerning quantum integrability*, J. Phys. A **19** (1986), no. 14, L841–L847. MR 857878 [MR0857878 \(87i:81100\)](#)
25. J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 1–42. MR 0082175 [MR0082175 \(18,511a\)](#)
26. C. Sevenheck and D. van Straten, *Deformation of singular Lagrangian subvarieties*, Math. Ann. **327** (2003), no. 1, 79–102. MR 2005122 [MR2005122 \(2004k:32046\)](#)
27. B. L. van der Waerden (ed.), *Sources of quantum mechanics*, Classics of science, vol. 5, Dover, 1968.
28. D. van Straten, *On the quantization problem*, 19 pp., 1991 (unpublished).
29. A. Weinstein, *Lagrangian submanifolds and Hamiltonian systems*, Ann. of Math. (2) **98** (1973), 377–410. MR 0331428 [MR0331428 \(48 #9761\)](#)

30. A. Weinstein and P. Xu, *Hochschild cohomology and characteristic classes for star-products*, Geometry of differential equations, Amer. Math. Soc. Transl. Ser. 2, vol. 186, Amer. Math. Soc., Providence, RI, 1998, pp. 177–194. MR 1732412 [MR1732412 \(2000m:53128\)](#)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

© Copyright American Mathematical Society 2011, 2012